Homework 7
Geometry

Joshua Ruiter

April 8, 2018

Proposition 0.1 (Exercise 8-13)**.** *There is a smooth vector field on S^2 that vanishes at exactly one point.*

Proof. Let N, S be the north and south poles of S^2 respectively. Let $\sigma : (S^2 \setminus \{N\}) \to \mathbb{R}^2$ be the stereographic projection and let $\widetilde{\sigma} : (S^2 \setminus \{S\}) \to \mathbb{R}^2$ be the corresponding projection omitting the south pole. Explicitly,

$$\sigma(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \qquad \sigma^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$
$$\widetilde{\sigma}(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) \qquad \widetilde{\sigma}^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right)$$

Note that the transition function $\sigma \circ \widetilde{\sigma}^{-1}$ is explicitly

$$\sigma \circ \widetilde{\sigma}^{-1}(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

We know that σ^{-1} is a diffeomorphism, because it is a smooth chart for S^2. By Proposition 8.19 (Lee), there is a unique smooth vector field $(\sigma^{-1})_*(\frac{\partial}{\partial x})$ on $S^2 \setminus \{N\}$. Define a rough vector field on all of S^2 by

$$X_p = \begin{cases} ((\sigma^{-1})_*(\frac{\partial}{\partial x}))_p & p \neq N \\ 0 & p = N \end{cases}$$

Because σ^{-1} is a diffeomorphism, $(\sigma^{-1})_*$ is a vector space isomorphism at each $p \in S^2 \setminus \{N\}$. Therefore, since $\frac{\partial}{\partial x}$ is nonzero everywhere, $(\sigma^{-1})_*$ does not vanish on $S^2 \setminus \{N\}$. We just need to show that X is smooth at the north pole.

If we compute the coordinate representation of X in the smooth chart $(S^2 \setminus \{S\}, \widetilde{\sigma})$, we will have another expression for X_p which is defined on $S^2 \setminus \{S\}$. This expression will agree for $p \in S^2 \setminus \{S\} \setminus \{N\}$ and will be smooth on $S^2 \setminus \{S\}$. One can check that the component functions $X^i(p)$ are zero at $p = N$. Thus X is smooth at N.

□

Lemma 0.2 (for Exercise 8-25)**.** *Let G be an abelian Lie group and let $i : G \to G$ be the inversion map $g \mapsto g^{-1}$. Then i is a Lie group isomorphism.*

Proof. The inversion map is smooth by definition of a Lie group. If $g, h \in G$, then

$$i(gh) = (gh)^{-1} = h^{-1}g^{-1} = i(h)i(g) = i(g)i(h)$$

using the fact that G is abelian. Thus i is a group homomorphism. It is injective since inverses are unique, and it is onto since every element has an inverse. Thus i is bijective. It is its own inverse, so it has a smooth inverse, so it is a diffeomorphism, so it is a Lie group isomorphism. □

Proposition 0.3 (Exercise 8-25)**.** *Let G be an abelian Lie group. Then the Lie algebra of G is abelian.*

Proof. Let $X, Y \in T_eG$ (that is, $X, Y \in \mathrm{Lie}(G)$). Using problem 7-2, we have

$$i_*X = -X \qquad i_*Y = -Y \qquad i_*[X, Y] = -(XY - YX)$$

Then we compute

$$[i_*X, i_*Y] = [-X, -Y] = (-X)(-Y) - (-Y)(-X) = XY - YX$$

Since the inversion map i is a Lie group homomorphism, $i_* : T_eG \to T_eG$ is a Lie algebra homomorphism (Theorem 8.44 of Lee) so

$$[X, Y] = XY - YX = [i_*X, i_*Y] = i_*[X, Y] = -(XY - YX) = -[X, Y]$$

Thus we have $[X, Y] = -[X, Y]$, which implies that $[X, Y] = 0$. Thus $\mathrm{Lie}(G)$ is abelian. □

Proposition 0.4 (Exercise 9-4)**.** *For $n \in \mathbb{N}$ we define a flow on $S^{2n-1} \subset \mathbb{C}^n$ by $\theta(t, z) = e^{it}z$. Then the infinitesimal generator of θ is a smooth non-vanishing vector field on S^{2n-1}.*

Proof. The infinitesimal generator of θ is the vector field V_z defined by

$$V_z = \frac{\partial}{\partial t} e^{it} z \bigg|_{t=0} = ie^{it}z \bigg|_{t=0} = iz$$

θ is smooth because it is essentially a map between Euclidean spaces, and its partial derivatives are all smooth. Since θ is smooth, by Proposition 9.11 V is a smooth vector field. We need to show that it is non-vanishing. By the above computation, $V_z = 0$ only if $z = 0$. But $z = 0 \notin S^{2n-1}$ so V is non-vanishing on S^{2n-1}. □

Lemma 0.5 (for Exercise 9-7)**.** *Let $B = B(0, 1)$ be the unit ball in $\mathbb{R}^n$ and let $p, q \in B$. There is a compactly supported smooth vector field X on B whose flow θ satisfies $\theta_1(p) = q$.*

Proof. Let

$$A = \{p + t(q - p) : 0 \le t \le 1\}$$

be the line segment connecting p, q. Note that A is closed. Because B is convex, $A \subset B$. Define a constant vector field X on A by $X_a = q - p$. Then X is trivially smooth. By Lemma 8.6 (Lee), there is a smooth vector field $\widetilde{X}$ on B such that $\widetilde{X}|_A = X$ and $\operatorname{supp} \widetilde{X} \subset B$. Thus $\widetilde{X}$ is a compactly supported smooth vector field on B. Note that

$$\gamma(t) = p + t(q - p)$$

is an integral curve of $\widetilde{X}$, as $\gamma'(t) = q - p = \widetilde{X}_{\gamma(t_0)}$ for any $0 \le t_0 \le 1$. Therefore, if θ is the flow of $\widetilde{X}$, we have $\theta_1(p) = \gamma(1) = q$. □

Proposition 0.6 (Exercise 9-7)**.** *Let M be a connected smooth manifold. Then the group of diffeomorphisms from M to itself acts transitively on M, that is, for $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$.*

Proof. Fix $p \in M$ and let U_p be the orbit of p under this action, that is,

$$U_p = \{q \in M : \exists F : M \to M \text{ such that } F(p) = q\}$$

where F is a diffeomorphism. First, note that U_p is non-empty, as the identity on M is a diffeomorphism, so $p \in U_p$. We claim that U_p is both open and closed. First, we show that U_p is open. Let $q \in U_p$ and let (V, ψ) be a smooth chart with $q \in V$ so that $\psi(V) = B(0,1) \subset \mathbb{R}^n$. We claim that $V \subset U_p$. Let $s \in V$. Then $\psi(s), \psi(q) \in B(0,1)$, so by the previous lemma, there is a compactly supported smooth vector field X on $B(0,1)$ with flow θ so that $\theta_1(\psi(q)) = \psi(s)$. By Proposition 8.19 (Lee), there is a unique smooth vector field Y on V that is ψ-related to X, that is,

$$\psi_* Y_r = X_r \quad \text{for } r \in V$$

Let η be the flow of Y. As Y is also compactly supported, by Lemma 8.6 (Lee) there is a smooth vector field $\widetilde{Y}$ on M such that $\widetilde{Y}|_{\operatorname{supp} Y} = Y$ and $\operatorname{supp} \widetilde{Y} \subset V$. If $\widetilde{\eta}$ is the flow of $\widetilde{Y}$, then $\widetilde{\eta}(t,p) = p$ outside V, so $\widetilde{\eta}$ is a smooth global extension of η. In particular, $\widetilde{\eta}_1$ i s a smooth extension of η_1, and $\widetilde{\eta}_1$ is a diffeomorphism on M. By Corollary 9.14 (Lee), as ψ^{-1} is a diffeomorphism,

$$\widetilde{\eta}_1 = \psi^{-1} \circ \theta_1 \circ \psi$$

In particular,

$$\widetilde{\eta}_1(q) = \psi^{-1} \circ \theta_1 \circ \psi(q) = \psi^{-1} \circ \psi(s) = s$$

By assumption, $q \in U_p$ so there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. Thus $\widetilde{\eta}_1 \circ F : M \to M$ is a diffeomorphism that maps p to s, so $s \in U_p$. Thus $V \subset U_p$, so U_p is open.

Now we show that U_p is closed. If $M \setminus U_p = \emptyset$, then we're done, so suppose $M \setminus U_p \neq \emptyset$. Let $s \in M \setminus U_p$ and let (V, ψ) be a smooth chart containing s with $\psi(V) = B(0,1)$. By the same reasoning as above, if V has non-empty intersection with U_p, say $r \in V \cap U_p$, then there is a diffeomorphism mapping p to r and a diffeomorphism mapping r to s, so $s \in U_p$, which is a contradiction. Thus $V \cap U_p = \emptyset$, so V is an open neighborhood of s contained in $M \setminus U_p$. Thus $M \setminus U_p$ is open, so U_p is closed.

We have shown that U_p is open, closed, and non-empty. Since M is connected, this implies that $U_p = M$. Thus there is only one orbit, so the action is transitive. □

Proposition 0.7 (Exercise 4a)**.** *Let X, Y, Z be the vector fields on $\mathbb{R}^3$ defined by*

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \qquad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \qquad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Define $\phi : \mathbb{R}^3 \to \mathfrak{X}(\mathbb{R}^3)$ by

$$(a, b, c) \mapsto aX + bY + cZ$$

Then ϕ is an isomorphism onto its image, and the bracket of vector fields on $\mathbb{R}^3$ corresponds to the cross product on $\mathbb{R}^3$. (That is, ϕ is a Lie algebra isomorphism onto its image.)

Proof. First we show that ϕ is linear. Let $\lambda \in \mathbb{R}$, and $(a, b, c), (d, e, f) \in \mathbb{R}^3$.

$$\begin{aligned}\phi(\lambda(a,b,c)+(d,e,f)) &= \phi(\lambda a+d, \lambda b+e, \lambda c+f) = (\lambda a+b)X+(\lambda b+e)Y+(\lambda c+f)Z \\ &= \lambda(aX+bY+cZ)+(dX+eY+fZ) = \lambda\phi(a,b,c)+\phi(d,e,f)\end{aligned}$$

Thus ϕ is linear. To show that ϕ is injective, we show that it has trivial kernel. If $(a, b, c) \in \ker\phi$, then

$$0 = aX+bY+cZ = (cy-bz)\frac{\partial}{\partial x}+(az-cx)\frac{\partial}{\partial y}+(bx-ay)\frac{\partial}{\partial z}$$

which implies that for all $x, y, z \in \mathbb{R}$, we have

$$cy-bz = az-cx = bx-ay = 0$$

In particular, this holds for $y = 1, z = 0$, so $c = 0$. Likewise, $z = 1, x = 0 \implies a = 0$, and $x = 1, y = 0 \implies b = 0$. Thus the kernel of ϕ is just $(0, 0, 0)$, so ϕ is injective. It is onto its image by definition, so it is an isomorphism onto its image.

Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$ (where $e_1 = (1, 0, 0)$, etc.). Then ϕ maps the basis $\{e_1, e_2, e_3\}$ to $\{X, Y, Z\}$ therefore $\{X, Y, Z\}$ is a basis for the image of ϕ (because ϕ is an isomorphism). The cross products of the standard basis are

$$e_1 \times e_2 = e_3 \qquad e_2 \times e_3 = e_1 \qquad e_3 \times e_1 = e_2$$

We also compute the brackets of our vector fields X, Y, Z, and find

$$[X,Y] = Z \qquad [Y,Z] = X \qquad [Z,X] = Y$$

Thus we have

$$\begin{aligned}\phi(e_1\times e_2) = \phi(e_3) = Z = [X,Y] = [\phi(e_1),\phi(e_2)] \\ \phi(e_2\times e_3) = \phi(e_1) = X = [Y,Z] = [\phi(e_2),\phi(e_3)] \\ \phi(e_3\times e_1) = \phi(e_2) = Y = [Z,X] = [\phi(e_3),\phi(e_1)]\end{aligned}$$

Thus the bracket of vector fields on $\mathbb{R}^3$ corresponds to the cross product on $\mathbb{R}^3$. In the language of Lie algebras, the cross product gives a Lie algebra structure to $\mathbb{R}^3$ by

$$\begin{aligned}[,] : \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \\ [x,y] &\mapsto x \times y\end{aligned}$$

and we have just shown that ϕ is a Lie algebra isomorphism onto its image. □

(Exercise 4b)
Compute the flow of $aX + bY + cZ$.
Solution. First, note that

$$aX+bY+cZ = (cy-bz)\frac{\partial}{\partial x}+(az-cx)\frac{\partial}{\partial y}+(bx-ay)\frac{\partial}{\partial z}$$

We compute the integral curves of this vector field by solving a system of ODEs. Let $\gamma(t) = (x(t), y(t), z(t))$ be an integral curve. Then we have

$$\begin{aligned}\dot{x} &= cy - bz\\ \dot{y} &= az - cx\\ \dot{z} &= bx - ay\end{aligned}$$

which we can also write as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let A be the 3-by-3 matrix above. The eigenvalues of A are $0, \pm\sqrt{a^2+b^2+c^2}$ as a routine calculation shows (compute the roots of $\det(A - \lambda I)$). Let $\lambda_1 = +\sqrt{a^2+b^2+c^2}$ and $\lambda_2 = -\sqrt{a^2+b^2+c^2}$. First we compute the eigenvector associated to $\lambda = 0$. We have

$$y = \frac{b}{a}x \qquad z = \frac{c}{a}x$$

so the eigenvector is $(1, \frac{b}{a}, \frac{c}{a})$ which is equivalent, up to scaling, to $v_0 = (a, b, c)$. (Note that we assume here that $a \neq 0$.) Now we compute the eigenvector associated to λ_1. We assume that $\lambda \neq 0$, so one of $a, b, c \neq 0$. WLOG, assume $a \neq 0$. We compute

$$\begin{aligned}y &= (1/\lambda)(az - cx) = (1/a)(bx - \lambda z)\\ z &= (1/\lambda)(bx - ay) = (1/a)(\lambda y + cx)\end{aligned}$$

so then

$$y = \frac{ab - \lambda c}{a^2 + \lambda^2}x \qquad z = \frac{ac + \lambda b}{a^2 + \lambda^2}x$$

so the associated eigenvector is

$$\left(1, \frac{ab - \lambda c}{a^2 + \lambda^2}, \frac{ac + \lambda b}{a^2 + \lambda^2}\right)x$$

which is equivalent up to rescaling to $(a^2 + \lambda^2, ab + \lambda c, \lambda b + ac)$. Note that when $\lambda = 0$ we recove the previous eigenvector $v_0 = (a, b, c)$. We denote by v_1 the eigenvector for λ_1 and v_2 the eigenvector for λ_2. The solutions to our system of ODEs then have the form

$$\gamma(t) = k_0 v_0 + k_1 e^{\lambda_1 t} v_1 + k_2 e^{\lambda_2 t} v_2$$

for scalars $k_0, k_1, k_2 \in \mathbb{R}$. So we know what all of the integral curves of $aX + bY + cZ$ look like. If θ is the flow of this vector field, then $\theta(t, p) = \gamma(t)$ where γ is an integral curve with $\gamma(0) = p$. To find k_0, k_1, k_2 so that $\gamma(0) = p = (x, y, z)$, we solve the linear system

$$\begin{bmatrix} v_0 & v_1 & v_2 \end{bmatrix}\begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & a^2 + \lambda_1^2 & a^2 + \lambda_2^2 \\ b & ab + \lambda_1 c & ab + \lambda_2 c \\ c & ac + \lambda_1 b & ac + \lambda_2 b \end{bmatrix}\begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Using Matlab, I computed the inverse of this 3×3 matrix, but it is excessively lengthy and complicated, so I don't include it here. Suffice it say, we can compute k_0, k_1, k_2 in terms of a, b, c and x, y, z so that the system is solved. Then we have

$$\gamma(0) = k_0 v_0 + k_1 v_1 + k_2 v_2 = (x, y, z)$$

and γ is an integral curve of $aX + bY + cZ$. Then the flow θ is given by

$$\theta(t, (x, y, z)) = \gamma(t)$$

keeping in mind that γ depends on k_0, k_1, k_2 which depend linearly on x, y, z.